RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 7 Solution

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November 6, 2014

Major criticism of CAPM theory: (1) assumptions about mean-variance analysis (everyone cares about mean and variance only) and (2) the existence of the market portfolio in which everyone invests.

1 What is Arbitrage?

(From Wikipedia) Arbitrage is possible when one of three conditions is met:

(1) The same asset does not trade at the same price on all markets.

(2) Two assets with identical cash flows do not trade at the same price.

(3) An asset with a known price in the future does not today trade at its future price discounted at the risk-free interest rate.

In each of the scenarios above, can you think of a strategy to make riskless profit?

Definition 1.1. *There is arbitrage if there exists a portfolio* Π*, whose value at time* t *is* Π_t*, satisfying for some* $T > 0$*,*

(a) (Initial cost of holding this portfolio is zero) $\Pi_0 = 0$*;*

(b) (Probability of resulting a loss is zero) $\mathbb{P}(\Pi_T < 0) = 0$*;*

(c) (Probability of resulting a gain is positive) $\mathbb{P}(\Pi_T > 0) > 0$*.*

2 Arbitrage Portfolio

Suppose we have a single factor model for the asset return

$$r_i = a_i + b_i F, \quad i = 1, 2, 3.$$

To form an arbitrage portfolio with weights w_i in asset i, we want

(1) Self-financing: Portfolio without any exogenous infusion or withdrawal of money. The purchase of a new asset must be financed by the sale of an old one.

$$w_1 + w_2 + w_3 = 0.$$

(2) Risk-free: In this factor model, the only risk to any asset comes from the factor F. To be risk-free, it suffices for the sensitivity of the portfolio $b_p = 0$. That is

$$b_1 w_1 + b_2 w_2 + b_3 w_3 = 0.$$

(3) Positive Return: Earning a positive return is the ultimate goal for the arbitrage portfolio:

$$\mu_1 w_1 + \mu_2 w_2 + \mu_3 w_3 > 0.$$

Example 2.1. Mr. Ting owns a portfolio with the following characteristics. Assume that returns are generated by a one-factor model.

Securities	**Factor-sensitivity**	**Proportion**	**Expected Return**
A	2	0.2	20%
B	3.50	0.4	10%
C	0.5	0.4	5%

Mr. Ting decides to create an arbitrage portfolio by increasing the holdings of security A by 0.2 (in proportion).

(a) What must be the weights of the other two securities in Mr. Ting's arbitrage portfolio?

(b) What is the expected return of the arbitrage portfolio?

Solution. (a) Denote the change of weights in B and C by X_B and X_C. Then $X_B + X_C = -0.2$. To create the arbitrage portfolio, we want to keep the risk unchanged. So the change in factor sensitivity of the portfolio should be 0. That is, $(0.2)(2) + 3.5X_B + 0.5X_C = 0$. Solving the two equations, we have $X_B = X_C = -0.1$.

(b) Expected return of the arbitrage portfolio =(0.2)(0.2) + (-0.1)(0.1) + (-0.1)(0.05) =0.025.

3 Arbitrage Pricing Theorem

Definition 3.1. *A simple factor model is a factor model without the error terms:*

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} F_j \quad \textit{for } i = 1, 2, \ldots, n$$

Theorem 3.1. *Suppose there are n assets whose returns are governed by m factors with $m < n$ according to the simple multi-factor model:*

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} F_j \quad \textit{for } i = 1, 2, \ldots, n$$

If no arbitrage exists, then there exists constants $\lambda_0, \lambda_1, \ldots, \lambda_m$ such that for each $i = 1, \ldots, n$,

$$\mu_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j$$

Proof. See P.209 in the textbook. □

Remark 3.1. If in addition to the factor terms, there are error terms:

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} F_j + e_i \quad \text{for } i = 1, 2, \ldots, n.$$

Consider the portfolio with weights $w_1, \ldots, w_n$. Then

$$r_p = a + \sum_{j=1}^{m} b_j F_j + e,$$

where $a = \sum_{i=1}^{n} w_i a_i, b_j = \sum_{i=1}^{n} w_i b_{ij}, \sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$. Then we can still apply the theorem if n is large, $\max_i \sigma_{e_i}^2 \leq S^2$ for some constant S and the portfolio is well diversified in the sense that for each i, there holds $w_i \leq W/n$ for some constant $W \simeq 1$ as the variance of the error term of a portfolio

$$\sigma_e^2 := \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 \leq \frac{1}{n^2} \sum_{i=1}^{n} W^2 S^2 \leq \frac{1}{n} W^2 S^2 \to 0,$$

as $n \to \infty$.

Example 3.1 (07-08 Final, with slight modification)**.** The returns r_i of three assets X_i satisfy the two-factor model

$$r_i = a_i + b_i f_1 + c_i f_2, \quad i = 1, 2, 3,$$

where b_i and c_i are factor loadings of risk factors f_1 and f_2 respectively. Consider the portfolio $X = \sum_{i=1}^{3} w_i X_i$, where it is assumed that $\sum_{i=1}^{3} w_i = 1$.

(a) What is the return r of the portfolio X?

(b) Evaluate $\mu := \mathrm{E}(r)$.

(c) Under the assumption of no arbitrage and APT, we know that $\mu = r_f$, where r_f denotes the risk-free rate. What can be said about $\sum_{i=1}^{3} w_i b_i$ and $\sum_{i=1}^{3} w_i c_i$? Please explain. Under these situations, further deduce that

$$\sum_{i=1}^{3} w_i a_i = r_f.$$

(d) Using the conditions derived in part (c), deduce that the weights w_i satisfy the matrix equation

$$\begin{pmatrix} a_1 - r_f & a_2 - r_f & a_3 - r_f \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

(e) Suppose that there exists weights $(w_1, w_2, w_3) \neq (0, 0, 0)$ satisfying (1). Assume further that the vectors (b_1, b_2, b_3) and (c_1, c_2, c_3) are linearly independent, show that there exists constants λ_1 and λ_2 satisfy the equation

$$a_i - r_f = \lambda_1 b_i + \lambda_2 c_i, \quad i = 1, 2, 3.$$

Hint: Is the matrix in (1) singular in this situation?

(f) Let the factor loadings be given by $(b_1, b_2, b_3) = (1, 0, 0)$ and $(c_1, c_2, c_3) = (0, 0, 1)$. Solve for λ_1 and λ_2. Further deduce that $a_2 = r_f$ in this case. What would be the weights (w_1, w_2, w_3) that satisfy (1) in this case?

Solution. (a) Since $X = \sum_{i=1}^{3} w_i X_i$,

$$r = \sum_{i=1}^{3} w_i r_i = \sum_{i=1}^{3} w_i(a_i + b_i f_1 + c_i f_2) = \sum_{i=1}^{3} w_i a_i + \left(\sum_{i=1}^{3} w_i b_i\right) f_1 + \left(\sum_{i=1}^{3} w_i c_i\right) f_2.$$

(b)

$$\mu = E[r] = \sum_{i=1}^{3} w_i a_i + \left(\sum_{i=1}^{3} w_i b_i\right) E(f_1) + \left(\sum_{i=1}^{3} w_i c_i\right) E(f_2).$$

(c) Since $\mu = r_f$, portfolio X must be the risk-free asset. This is because if X contains risk, it should provide a higher expected return than the risk-free rate at equilibrium as investors are risk-averse. Therefore X has zero sensitivity to both factor 1 and 2, and from the result of (a), we have

$$\sum_{i=1}^{3} w_i b_i = \sum_{i=1}^{3} w_i c_i = 0.$$

Hence, we have $\displaystyle\sum_{i=1}^{3} w_i a_i = \sum_{i=1}^{3} w_i a_i + 0 \times E(f_1) + 0 \times E(f_2) = \mu = r_f$.

(d) Note that $\displaystyle\sum_{i=1}^{3} w_i = 1 \Rightarrow \sum_{i=1}^{3} w_i a_i = r_f = \left(\sum_{i=1}^{3} w_i\right) r_f \Rightarrow \sum_{i=1}^{3} (a_i - r_f) w_i = 0$.

Combining with $\displaystyle\sum_{i=1}^{3} w_i b_i = \sum_{i=1}^{3} w_i c_i = 0$, the result follows.

(e) If there exist a non-trivial solution $(w_1, w_2, w_3) \neq (0,0,0)$ satisfying the linear system, the coefficient matrix on the LHS must be singular and hence the three row vectors are linearly dependent. By definition of linear dependence, there exist a non-zero vector (y_0, y_1, y_2) such that

$$y_0(a_1 - r_f, a_2 - r_f, a_3 - r_f) + y_1(b_1, b_2, b_3) + y_2(c_1, c_2, c_3) = (0,0,0).$$

Note that $y_0 \neq 0$. If it were 0, $y_1(b_1, b_2, b_3) + y_2(c_1, c_2, c_3) = (0,0,0)$ will imply $y_1 = y_2 = 0$ by the linear independence of $\boldsymbol{b}$ and $\boldsymbol{c}$, which contradicts with the fact that (y_0, y_1, y_2) is not the zero vector. Thus,

$$a_i - r_f = -\frac{y_1}{y_0} b_i - \frac{y_2}{y_0} c_i = \lambda_1 b_i + \lambda_2 c_i, \text{ for } i = 1, 2, 3,$$

where $\lambda_1 := -\dfrac{y_1}{y_0}, \lambda_2 := -\dfrac{y_2}{y_0}$.

(f) Substituting $(b_1, b_2, b_3) = (1, 0, 0)$ and $(c_1, c_2, c_3) = (0, 0, 1)$, we have

$$\begin{aligned} \lambda_1 &= a_1 - r_f \\ \lambda_2 &= a_3 - r_f \\ a_2 - r_f &= 0. \end{aligned}$$

Hence, $a_2 = r_f$. Note that equation (1) becomes

$$\begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $w_1 = w_3 = 0$ and w_2 is arbitrary. Using the constraint that $\sum_{i=1}^{3} w_i = 1$, we have $(w_1, w_2, w_3) = (0, 1, 0)$.